

# Unbounded subnormal weighted shifts on directed trees

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**ABSTRACT.** A new method of verifying the subnormality of unbounded Hilbert space operators based on an approximation technique is proposed. Diverse sufficient conditions for subnormality of unbounded weighted shifts on directed trees are established. An approach to this issue via consistent systems of probability measures is invented. The role played by determinate Stieltjes moment sequences is elucidated. Lambert's characterization of subnormality of bounded operators is shown to be valid for unbounded weighted shifts on directed trees that have sufficiently many quasi-analytic vectors, which is a new phenomenon in this area.

## 1. Introduction

The theory of bounded subnormal operators was originated by P. Halmos in [16]. Nowadays, its foundations are well-developed (see [9]; see also [10] for a recent survey article on this subject). The theory of unbounded symmetric operators had been established much earlier (see [52] and the monograph [45]). In view of Naimark's theorem, these operators are particular cases of *unbounded* subnormal operators, i.e., densely defined operators having normal extensions in (possibly larger) Hilbert spaces. The first general results on unbounded subnormal operators appeared in [5] and [15] (see also [38]). A systematic study of this class of operators was undertaken in the trilogy [41, 42, 43]. The theory of unbounded subnormal operators has intimate connections with other branches of mathematics and quantum physics (see [49, 6, 2] and [20, 40, 48, 21]). It has been developed in two main directions, the first is purely theoretical (cf. [26, 44, 14, 12, 13, 51, 1]), the other is related to special classes of operators (cf. [11, 22, 23, 24]). In this paper, we will focus mostly on the class of weighted shifts on directed trees.

The notion of a weighted shift on a directed tree generalizes that of a weighted shift on the  $\ell^2$  space, the classical object of operator theory (see [28, 35, 27]). In a recent paper [17] some fundamental properties of weighted shifts on directed trees have been studied. Although considerable progress has been made in this field, a

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number of important questions have not been answered. In this paper we continue investigations along these lines with special emphasis put on the issue of subnormality of unbounded operators, the case which is essentially more complicated and not an easy extension of the bounded one. The main difficulty comes from the fact that the celebrated Lambert characterization of subnormality of bounded operators (cf. [25]) is no longer valid for unbounded ones (see Section 3.2; see also [18] for a surprising counterexample). A new criterion (read: sufficient condition) for subnormality of unbounded operators has been invented recently in [7]. By using it, we will show that subnormality is preserved by a certain weak-type limit procedure (see Theorem 3.1.1). This enables us to perform the approximation process relevant to unbounded weighted shifts on directed trees. What we get is Theorem 5.1.1, which is the main result of this paper (its proof depends essentially on the passage through weighted shifts that may have zero weights). It provides a criterion for subnormality of unbounded weighted shifts on directed trees written in terms of consistent systems of measures, which is new even in the case of bounded operators. Roughly speaking, for bounded and some unbounded operators, the assumption that  $C^\infty$ -vectors generates Stieltjes moment sequences implies subnormality (the reverse implication is always true, cf. Proposition 3.2.1). As discussed in Section 3.2, there are unbounded formally normal operators having dense set of  $C^\infty$ -vectors, for which this is not true. It is a surprising fact that there are non-hyponormal operators having dense set of  $C^\infty$ -vectors generating Stieltjes moment sequences. These are carefully constructed weighted shifts on a leafless directed tree with one branching vertex (cf. [18]). They do not satisfy the consistency condition 2° of Lemma 4.1.3 and none of them has consistent system of measures.

Under some additional assumption, the criterion for subnormality formulated in Theorem 5.1.1 becomes a full characterization (cf. Theorem 5.1.3). This is the case in the presence of quasi-analytic vectors (cf. Theorem 5.3.1), which is the first result of this kind (see Section 5.3 for more comments).

## 2. Preliminaries

**2.1. Notation and terminology.** Let  $\mathbb{R}$  and  $\mathbb{C}$  stand for the sets of real and complex numbers respectively. Define

$$\mathbb{Z}_+ = \{0, 1, 2, 3, \dots\}, \mathbb{N} = \{1, 2, 3, 4, \dots\} \text{ and } \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}.$$

We write  $\mathfrak{B}(\mathbb{R}_+)$  for the  $\sigma$ -algebra of all Borel subsets of  $\mathbb{R}_+$ . The closed support of a positive Borel measure  $\mu$  on  $\mathbb{R}_+$  is denoted by  $\text{supp } \mu$ . We write  $\delta_0$  for the Borel probability measure on  $\mathbb{R}_+$  concentrated at 0.

Let  $A$  be an operator in a complex Hilbert space  $\mathcal{H}$  (all operators considered in this paper are linear). Denote by  $\mathcal{D}(A)$  and  $A^*$  the domain and the adjoint of  $A$  (in case it exists). Set  $\mathcal{D}^\infty(A) = \bigcap_{n=0}^\infty \mathcal{D}(A^n)$ ; members of  $\mathcal{D}^\infty(A)$  are called  $C^\infty$ -vectors of  $A$ . A linear subspace  $\mathcal{E}$  of  $\mathcal{D}(A)$  is said to be a *core* of  $A$  if the graph of  $A$  is contained in the closure of the graph of the restriction  $A|_{\mathcal{E}}$  of  $A$  to  $\mathcal{E}$ . If  $A$  is closed, then  $\mathcal{E}$  is a core of  $A$  if and only if  $A$  coincides with the closure of  $A|_{\mathcal{E}}$ . A closed densely defined operator  $N$  in  $\mathcal{H}$  is said to be *normal* if  $N^*N = NN^*$  (equivalently:  $\mathcal{D}(N) = \mathcal{D}(N^*)$  and  $\|N^*h\| = \|Nh\|$  for all  $h \in \mathcal{D}(N)$ ). For other facts concerning unbounded operators (including normal ones) that are needed in this paper we refer the reader to [4, 53]. A densely defined operator  $S$  in  $\mathcal{H}$  is said to be *subnormal* if there exists a complex Hilbert space  $\mathcal{K}$  and a normal operator

$N$  in  $\mathcal{K}$  such that  $\mathcal{H} \subseteq \mathcal{K}$  (isometric embedding) and  $Sh = Nh$  for all  $h \in \mathcal{D}(S)$ . It is clear that subnormal operators are closable and their closures are subnormal.

In what follows,  $\mathcal{B}(\mathcal{H})$  stands for the  $C^*$ -algebra of all bounded operators  $A$  in  $\mathcal{H}$  such that  $\mathcal{D}(A) = \mathcal{H}$ . We write  $\text{LIN } \mathcal{F}$  for the linear span of a subset  $\mathcal{F}$  of  $\mathcal{H}$ .

**2.2. Directed trees.** Let  $\mathcal{T} = (V, E)$  be a directed tree ( $V$  and  $E$  stand for the sets of vertices and edges of  $\mathcal{T}$ , respectively). If  $\mathcal{T}$  has a root, which will be denoted by  $\text{root}$ , then we write  $V^\circ := V \setminus \{\text{root}\}$ ; otherwise, we put  $V^\circ = V$ . Set  $\text{Chi}(u) = \{v \in V : (u, v) \in E\}$  for  $u \in V$ . A member of  $\text{Chi}(u)$  is called a *child* of  $u$ . For every vertex  $u \in V^\circ$  there exists a unique vertex, denoted by  $\text{par}(u)$ , such that  $(\text{par}(u), u) \in E$ . The correspondence  $u \mapsto \text{par}(u)$  is a partial function from  $V$  to  $V$ . For  $n \in \mathbb{N}$ , the  $n$ -fold composition of the partial function  $\text{par}$  with itself will be denoted by  $\text{par}^n$ . Let  $\text{par}^0$  stand for the identity map on  $V$ . We call  $\mathcal{T}$  *leafless* if  $V = V'$ , where  $V' := \{u \in V : \text{Chi}(u) \neq \emptyset\}$ .

It is well-known that (see e.g., [17, Proposition 2.1.2])  $\text{Chi}(u) \cap \text{Chi}(v) = \emptyset$  for all  $u, v \in V$  such that  $u \neq v$ , and

$$(2.2.1) \quad V^\circ = \bigsqcup_{u \in V} \text{Chi}(u).$$

(The symbol “ $\bigsqcup$ ” denotes disjoint union of sets.) For a subset  $W \subseteq V$ , we put  $\text{Chi}(W) = \bigsqcup_{v \in W} \text{Chi}(v)$  and define  $\text{Chi}^{(0)}(W) = W$ ,  $\text{Chi}^{(n+1)}(W) = \text{Chi}(\text{Chi}^{(n)}(W))$  for  $n \in \mathbb{Z}_+$  and  $\text{Des}(W) = \bigcup_{n=0}^{\infty} \text{Chi}^{(n)}(W)$ . By induction, we have

$$(2.2.2) \quad \text{Chi}^{(n+1)}(W) = \bigcup_{v \in \text{Chi}(W)} \text{Chi}^{(n)}(\{v\}), \quad n \in \mathbb{Z}_+,$$

$$(2.2.3) \quad \text{Chi}^{(m)}(\text{Chi}^{(n)}(W)) = \text{Chi}^{(m+n)}(W), \quad m, n \in \mathbb{Z}_+.$$

We shall abbreviate  $\text{Chi}^{(n)}(\{u\})$  and  $\text{Des}(\{u\})$  to  $\text{Chi}^{(n)}(u)$  and  $\text{Des}(u)$  respectively. We now state some useful properties of the functions  $\text{Chi}^{(n)}(\cdot)$  and  $\text{Des}(\cdot)$ .

**PROPOSITION 2.2.1.** *If  $\mathcal{T}$  is a directed tree, then*

$$(2.2.4) \quad \text{Chi}^{(n)}(u) = \{w \in V : \text{par}^n(w) = u\}, \quad n \in \mathbb{Z}_+, u \in V,$$

$$(2.2.5) \quad \text{Chi}^{(n+1)}(u) = \bigsqcup_{v \in \text{Chi}(u)} \text{Chi}^{(n)}(v), \quad n \in \mathbb{Z}_+, u \in V,$$

$$(2.2.6) \quad \text{Chi}^{(n+1)}(u) = \bigsqcup_{v \in \text{Chi}^{(n)}(u)} \text{Chi}(v), \quad n \in \mathbb{Z}_+, u \in V,$$

$$(2.2.7) \quad \text{Des}(u) = \bigsqcup_{n=0}^{\infty} \text{Chi}^{(n)}(u), \quad u \in V,$$

$$(2.2.8) \quad \text{Des}(u_1) \cap \text{Des}(u_2) = \emptyset, \quad u_1, u_2 \in \text{Chi}(u), u_1 \neq u_2, u \in V.$$

**PROOF.** Equality (2.2.4) follows by induction on  $n$ . Combining (2.2.2) with the fact that the sets  $\text{Chi}^{(n)}(u)$ ,  $u \in V$ , are pairwise disjoint for every fixed integer  $n \geq 0$ , we get (2.2.5). Equality (2.2.6) follows from the definition of  $\text{Chi}^{(n+1)}(u)$  and (2.2.1). Using the definition of  $\text{par}$  and the fact that  $\mathcal{T}$  has no circuits, we deduce that the sets  $\text{Chi}^{(n)}(u)$ ,  $n \in \mathbb{Z}_+$ , are pairwise disjoint. Hence, (2.2.7) holds. Assertion (2.2.8) can be deduced from (2.2.4) and (2.2.7).  $\square$

**2.3. Weighted shifts on directed trees.** In what follows, given a directed tree  $\mathcal{T}$ , we tacitly assume that  $V$  and  $E$  stand for the sets of vertices and edges of  $\mathcal{T}$  respectively. Denote by  $\ell^2(V)$  the Hilbert space of all square summable complex functions on  $V$  with the standard inner product  $\langle f, g \rangle = \sum_{u \in V} f(u) \overline{g(u)}$ . For  $u \in V$ , we define  $e_u \in \ell^2(V)$  to be the characteristic function of the one-point set  $\{u\}$ . Then  $\{e_u\}_{u \in V}$  is an orthonormal basis of  $\ell^2(V)$ . Set  $\mathcal{E}_V = \text{LIN}\{e_u : u \in V\}$ .

For  $\lambda = \{\lambda_v\}_{v \in V^\circ} \subseteq \mathbb{C}$ , we define the operator  $S_\lambda$  in  $\ell^2(V)$  by

$$\begin{aligned} \mathcal{D}(S_\lambda) &= \{f \in \ell^2(V) : \Lambda_{\mathcal{T}} f \in \ell^2(V)\}, \\ S_\lambda f &= \Lambda_{\mathcal{T}} f, \quad f \in \mathcal{D}(S_\lambda), \end{aligned}$$

where  $\Lambda_{\mathcal{T}}$  is the mapping defined on functions  $f : V \rightarrow \mathbb{C}$  via

$$(2.3.1) \quad (\Lambda_{\mathcal{T}} f)(v) = \begin{cases} \lambda_v \cdot f(\text{par}(v)) & \text{if } v \in V^\circ, \\ 0 & \text{if } v = \text{root}. \end{cases}$$

We call  $S_\lambda$  a *weighted shift* on the directed tree  $\mathcal{T}$  with weights  $\lambda = \{\lambda_v\}_{v \in V^\circ}$ .

Given a family  $\{\lambda_v\}_{v \in V^\circ} \subseteq \mathbb{C}$ , we define the family  $\{\lambda_{u|v}\}_{u \in V, v \in \text{Des}(u)}$  by

$$(2.3.2) \quad \lambda_{u|v} = \begin{cases} 1 & \text{if } v = u, \\ \prod_{j=0}^{n-1} \lambda_{\text{par}^j(v)} & \text{if } v \in \text{Chi}^{(n)}(u), n \geq 1. \end{cases}$$

Note that due to (2.2.7) the above definition is correct and

$$(2.3.3) \quad \lambda_{u|w} = \lambda_{u|v} \lambda_w, \quad w \in \text{Chi}(v), v \in \text{Des}(u), u \in V,$$

$$(2.3.4) \quad \lambda_{\text{par}(v)|w} = \lambda_v \lambda_{v|w}, \quad v \in V^\circ, w \in \text{Des}(v).$$

The following lemma is a generalization of [17, Lemma 6.1.1] to the case of unbounded operators. From now on, we adopt the convention that  $\sum_{v \in \emptyset} x_v = 0$ .

**LEMMA 2.3.1.** *Let  $S_\lambda$  be a weighted shift on a directed tree  $\mathcal{T}$  with weights  $\lambda = \{\lambda_v\}_{v \in V^\circ}$ . Fix  $u \in V$  and  $n \in \mathbb{Z}_+$ . Then the following assertions hold:*

- (i)  $e_u \in \mathcal{D}(S_\lambda^n)$  if and only if  $\sum_{v \in \text{Chi}^{(m)}(u)} |\lambda_{u|v}|^2 < \infty$  for all integers  $m$  such that  $1 \leq m \leq n$ ,
- (ii) if  $e_u \in \mathcal{D}(S_\lambda^n)$ , then  $S_\lambda^n e_u = \sum_{v \in \text{Chi}^{(n)}(u)} \lambda_{u|v} e_v$ ,
- (iii) if  $e_u \in \mathcal{D}(S_\lambda^n)$ , then  $\|S_\lambda^n e_u\|^2 = \sum_{v \in \text{Chi}^{(n)}(u)} |\lambda_{u|v}|^2$ .

**PROOF.** For  $k \in \mathbb{Z}_+$ , we define the complex function  $\lambda_{u|}^{(k)}$  on  $V$  by

$$(2.3.5) \quad \lambda_{u|v}^{(k)} = \begin{cases} \lambda_{u|v} & \text{if } v \in \text{Chi}^{(k)}(u), \\ 0 & \text{if } v \in V \setminus \text{Chi}^{(k)}(u). \end{cases}$$

We shall prove that for every  $k \in \mathbb{Z}_+$ , the following two conditions hold

$$(2.3.6) \quad e_u \in \mathcal{D}(S_\lambda^k) \text{ if and only if } \sum_{v \in \text{Chi}^{(m)}(u)} |\lambda_{u|v}|^2 < \infty \text{ for } m = 0, 1, \dots, k,$$

$$(2.3.7) \quad \text{if } e_u \in \mathcal{D}(S_\lambda^k), \text{ then } S_\lambda^k e_u = \lambda_{u|}^{(k)}.$$

We use an induction on  $k$ . The case of  $k = 0$  is obvious. Suppose that (2.3.6) and (2.3.7) hold for all nonnegative integers less than or equal to  $k$ . Assume that

$e_u \in \mathcal{D}(S_{\lambda}^k)$ . Now we compute  $\Lambda_{\mathcal{T}}(S_{\lambda}^k e_u)$ . It follows from the induction hypothesis and (2.3.5) that

$$\begin{aligned} (\Lambda_{\mathcal{T}}(S_{\lambda}^k e_u))(v) &\stackrel{(2.3.1)}{=} \begin{cases} \lambda_v(S_{\lambda}^k e_u)(\text{par}(v)) & \text{if } v \in V^{\circ}, \\ 0 & \text{if } v = \text{root}, \end{cases} \\ &\stackrel{(2.3.7)}{=} \begin{cases} \lambda_v \lambda_{u|\text{par}(v)}^{\langle k \rangle} & \text{if } \text{par}(v) \in \text{Chi}^{\langle k \rangle}(u), \\ 0 & \text{otherwise,} \end{cases} \\ &\stackrel{(2.2.4)}{=} \begin{cases} \lambda_v \lambda_{u|\text{par}(v)} & \text{if } v \in \text{Chi}^{\langle k+1 \rangle}(u), \\ 0 & \text{otherwise,} \end{cases} \\ &\stackrel{(2.3.3)}{=} \lambda_{u|v}^{\langle k+1 \rangle}, \quad v \in V, \end{aligned}$$

which shows that  $\Lambda_{\mathcal{T}}(S_{\lambda}^k e_u) = \lambda_{u|}^{\langle k+1 \rangle}$ . This in turn implies that (2.3.6) and (2.3.7) hold for  $k+1$  in place of  $k$ . This proves (i) and (ii). Assertion (iii) is a direct consequence of (ii).  $\square$

The following result is an essential ingredient of the proof of Theorem 5.1.1.

**PROPOSITION 2.3.2.** *If  $\lambda^{(i)} = \{\lambda_v^{(i)}\}_{v \in V^{\circ}}$ ,  $i = 1, 2, 3, \dots$ , and  $\lambda = \{\lambda_v\}_{v \in V^{\circ}}$  are families of complex numbers such that*

- (i)  $\mathcal{E}_V \subseteq \mathcal{D}^{\infty}(S_{\lambda}) \cap \bigcap_{i=1}^{\infty} \mathcal{D}^{\infty}(S_{\lambda^{(i)}})$ ,
- (ii)  $\lim_{i \rightarrow \infty} \lambda_v^{(i)} = \lambda_v$  for all  $v \in V^{\circ}$ ,
- (iii)  $\lim_{i \rightarrow \infty} \|S_{\lambda^{(i)}}^n e_u\| = \|S_{\lambda}^n e_u\|$  for all  $n \in \mathbb{Z}_+$  and  $u \in V$ ,

then

$$(2.3.8) \quad \langle S_{\lambda}^m e_u, S_{\lambda}^n e_v \rangle = \lim_{i \rightarrow \infty} \langle S_{\lambda^{(i)}}^m e_u, S_{\lambda^{(i)}}^n e_v \rangle, \quad u, v \in V, m, n \in \mathbb{Z}_+.$$

**PROOF.** We split the proof into two steps.

**STEP 1.** If  $\lambda = \{\lambda_v\}_{v \in V^{\circ}}$  is a family of complex numbers such that  $\mathcal{E}_V \subseteq \mathcal{D}^{\infty}(S_{\lambda})$ , then for all  $m, n \in \mathbb{Z}_+$  and  $u, v \in V$ ,

$$(2.3.9) \quad \langle S_{\lambda}^m e_u, S_{\lambda}^n e_v \rangle = \begin{cases} 0 & \text{if } \mathcal{C}^{m,n}(u, v) = \emptyset, \\ \overline{\lambda_{v|u}} \|S_{\lambda}^m e_u\|^2 & \text{if } \mathcal{C}^{m,n}(u, v) \neq \emptyset \text{ and } m \leq n, \\ \lambda_{u|v} \|S_{\lambda}^n e_v\|^2 & \text{if } \mathcal{C}^{m,n}(u, v) \neq \emptyset \text{ and } m > n, \end{cases}$$

where  $\mathcal{C}^{m,n}(u, v) := \text{Chi}^{\langle m \rangle}(u) \cap \text{Chi}^{\langle n \rangle}(v)$ .

Indeed, it follows from Lemma 2.3.1 that

$$\begin{aligned} (2.3.10) \quad \langle S_{\lambda}^m e_u, S_{\lambda}^n e_v \rangle &= \left\langle \sum_{u' \in \text{Chi}^{\langle m \rangle}(u)} \lambda_{u|u'} e_{u'}, \sum_{v' \in \text{Chi}^{\langle n \rangle}(v)} \lambda_{v|v'} e_{v'} \right\rangle \\ &= \sum_{u' \in \mathcal{C}^{m,n}(u, v)} \lambda_{u|u'} \overline{\lambda_{v|u'}}. \end{aligned}$$

Hence, if  $\mathcal{C}^{m,n}(u, v) = \emptyset$ , then the left-hand side of (2.3.9) is equal to 0 as required. Suppose now that  $\mathcal{C}^{m,n}(u, v) \neq \emptyset$  and  $m \leq n$ . Then

$$(2.3.11) \quad \mathcal{C}^{m,n}(u, v) = \text{Chi}^{\langle m \rangle}(u).$$

To show this, take  $w \in \mathcal{C}^{m,n}(u, v)$ . Then, by (2.2.4),  $u = \text{par}^m(w)$  and

$$v = \text{par}^n(w) = \text{par}^{n-m}(\text{par}^m(w)) = \text{par}^{n-m}(u),$$

which, by (2.2.4) again, is equivalent to

$$(2.3.12) \quad u \in \text{Chi}^{\langle n-m \rangle}(v).$$

This implies that

$$(2.3.13) \quad \text{Chi}^{\langle m \rangle}(u) \subseteq \text{Chi}^{\langle m \rangle}(\text{Chi}^{\langle n-m \rangle}(v)) \stackrel{(2.2.3)}{=} \text{Chi}^{\langle n \rangle}(v).$$

Thus (2.3.11) holds. Next, we show that

$$(2.3.14) \quad \lambda_{v|u'} = \lambda_{u|u'} \lambda_{v|u}, \quad u' \in \text{Chi}^{\langle m \rangle}(u).$$

It is enough to consider the case where  $m \geq 1$  and  $n > m$ . Since  $u' \in \text{Chi}^{\langle m \rangle}(u)$ , we infer from (2.3.13) that  $u' \in \text{Chi}^{\langle n \rangle}(v)$ . Moreover, by (2.3.12),  $u \in \text{Chi}^{\langle n-m \rangle}(v)$ . All these facts together with (2.3.2) imply that

$$\begin{aligned} \lambda_{v|u'} &= \prod_{j=0}^{n-1} \lambda_{\text{par}^j(u')} = \prod_{j=0}^{m-1} \lambda_{\text{par}^j(u')} \prod_{j=m}^{n-1} \lambda_{\text{par}^j(u')} \\ &\stackrel{(2.3.2)}{=} \lambda_{u|u'} \prod_{j=0}^{n-m-1} \lambda_{\text{par}^j(\text{par}^m(u'))} \stackrel{(2.2.4)}{=} \lambda_{u|u'} \prod_{j=0}^{n-m-1} \lambda_{\text{par}^j(u)} \stackrel{(2.3.2)}{=} \lambda_{u|u'} \lambda_{v|u}, \end{aligned}$$

which completes the proof of (2.3.14). Now applying (2.3.10), (2.3.11), (2.3.14) and Lemma 2.3.1 (iii), we obtain

$$\begin{aligned} \langle S_{\lambda}^m e_u, S_{\lambda}^n e_v \rangle &= \sum_{u' \in \text{Chi}^{\langle m \rangle}(u)} \lambda_{u|u'} \overline{\lambda_{v|u'}} \\ &\stackrel{(2.3.14)}{=} \overline{\lambda_{v|u}} \sum_{u' \in \text{Chi}^{\langle m \rangle}(u)} |\lambda_{u|u'}|^2 = \overline{\lambda_{v|u}} \|S_{\lambda}^m e_u\|^2. \end{aligned}$$

Taking the complex conjugate and making appropriate substitutions, we infer from the above that  $\langle S_{\lambda}^m e_u, S_{\lambda}^n e_v \rangle = \lambda_{u|v} \|S_{\lambda}^n e_v\|^2$  if  $\mathcal{C}^{m,n}(u, v) \neq \emptyset$  and  $m > n$ , which completes the proof of Step 1.

STEP 2. Under the assumptions of Proposition 2.3.2, equality (2.3.8) holds.

Indeed, it follows from (ii) that

$$(2.3.15) \quad \lim_{i \rightarrow \infty} \lambda_{u|v}^{(i)} = \lambda_{u|v}, \quad u \in V, v \in \text{Des}(u),$$

where  $\{\lambda_{u|v}^{(i)}\}_{u \in V, v \in \text{Des}(u)}$  is the family related to  $\{\lambda_v^{(i)}\}_{v \in V^\circ}$  via (2.3.2). Now, applying Step 1 to the operators  $S_{\lambda^{(i)}}$  and  $S_{\lambda}$  (which is possible due to (i)) and using (2.3.15) and (iii), we obtain (2.3.8).  $\square$

**2.4. Backward extensions of Stieltjes moment sequences.** We say that a sequence  $\{t_n\}_{n=0}^\infty$  of real numbers is a *Stieltjes moment sequence* if there exists a positive Borel measure  $\mu$  on  $\mathbb{R}_+$  such that

$$t_n = \int_0^\infty s^n d\mu(s), \quad n \in \mathbb{Z}_+,$$

where  $\int_0^\infty$  means integration over the set  $\mathbb{R}_+$ ;  $\mu$  is called a *representing measure* of  $\{t_n\}_{n=0}^\infty$ . A Stieltjes moment sequence is said to be *determinate* if it has only one representing measure. By the Stieltjes theorem (cf. [36, Theorem 1.3] or [3, Theorem 6.2.5]), a sequence  $\{t_n\}_{n=0}^\infty \subseteq \mathbb{R}$  is a Stieltjes moment sequence if and only if the sequences  $\{t_n\}_{n=0}^\infty$  and  $\{t_{n+1}\}_{n=0}^\infty$  are positive definite (recall that a

sequence  $\{t_n\}_{n=0}^\infty \subseteq \mathbb{R}$  is said to be *positive definite* if  $\sum_{k,l=0}^n t_{k+l} \alpha_k \overline{\alpha_l} \geq 0$  for all  $\alpha_0, \dots, \alpha_n \in \mathbb{C}$  and  $n \in \mathbb{Z}_+$ . It is clear from the definition that

$$(2.4.1) \quad \text{if } \{t_n\}_{n=0}^\infty \text{ is a Stieltjes moment sequence, then so is } \{t_{n+1}\}_{n=0}^\infty.$$

The converse is not true in general. Moreover, if  $\{t_n\}_{n=0}^\infty$  is an indeterminate Stieltjes moment sequence, then so is  $\{t_{n+1}\}_{n=0}^\infty$  (see Lemma 2.4.1 or [37, Proposition 5.12]). The converse implication fails to hold (cf. [37, Corollary 4.21]; see also [18]).

The question of backward extendibility of Hamburger moment sequences has well-known solutions (see e.g., [54] and [47]). Below, we formulate a solution of a variant of this question for Stieltjes moment sequences (see [17, Lemma 6.1.2] for the special case of compactly supported representing measures).

LEMMA 2.4.1. *Let  $\{t_n\}_{n=0}^\infty$  be a Stieltjes moment sequence and let  $\vartheta$  be a positive real number. Set  $t_{-1} = \vartheta$ . Then the following are equivalent:*

- (i)  $\{t_{n-1}\}_{n=0}^\infty$  is a Stieltjes moment sequence,
- (ii)  $\{t_{n-1}\}_{n=0}^\infty$  is positive definite,
- (iii) there is a representing measure  $\mu$  of  $\{t_n\}_{n=0}^\infty$  such that<sup>1</sup>  $\int_0^\infty \frac{1}{s} d\mu(s) \leq \vartheta$ .

Moreover, if (i) holds, then the mapping  $\mathcal{M}_0(\vartheta) \ni \mu \rightarrow \nu_\mu \in \mathcal{M}_{-1}(\vartheta)$  defined by

$$(2.4.2) \quad \nu_\mu(\sigma) = \int_\sigma \frac{1}{s} d\mu(s) + \left( \vartheta - \int_0^\infty \frac{1}{s} d\mu(s) \right) \delta_0(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),$$

is a bijection with the inverse  $\mathcal{M}_{-1}(\vartheta) \ni \nu \rightarrow \mu_\nu \in \mathcal{M}_0(\vartheta)$  given by

$$(2.4.3) \quad \mu_\nu(\sigma) = \int_\sigma s d\nu(s), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),$$

where  $\mathcal{M}_0(\vartheta)$  stands for the set of all representing measures  $\mu$  of  $\{t_n\}_{n=0}^\infty$  such that  $\int_0^\infty \frac{1}{s} d\mu(s) \leq \vartheta$ , and  $\mathcal{M}_{-1}(\vartheta)$  for the set of all representing measures  $\nu$  of  $\{t_{n-1}\}_{n=0}^\infty$ . In particular,  $\nu_\mu(\{0\}) = 0$  if and only if  $\int_0^\infty \frac{1}{s} d\mu(s) = \vartheta$ .

If (i) holds and  $\{t_n\}_{n=0}^\infty$  is determinate, then  $\{t_{n-1}\}_{n=0}^\infty$  is determinate, the unique representing measure  $\mu$  of  $\{t_n\}_{n=0}^\infty$  satisfies the inequality  $\int_0^\infty \frac{1}{s} d\mu(s) \leq \vartheta$ , and  $\nu_\mu$  is the unique representing measure of  $\{t_{n-1}\}_{n=0}^\infty$ .

PROOF. Equivalence (i) $\Leftrightarrow$ (ii) follows from the Stieltjes theorem.

(iii) $\Rightarrow$ (i) Clearly, if  $\mu \in \mathcal{M}_0(\vartheta)$ , then  $t_{n-1} = \int_0^\infty s^n d\nu_\mu(s)$  for all  $n \in \mathbb{Z}_+$ , which means that  $\{t_{n-1}\}_{n=0}^\infty$  is a Stieltjes moment sequence and  $\nu_\mu \in \mathcal{M}_{-1}(\vartheta)$ .

(i) $\Rightarrow$ (iii) Take  $\nu \in \mathcal{M}_{-1}(\vartheta)$ . Setting  $\mu := \mu_\nu$  (cf. (2.4.3)), we see that

$$(2.4.4) \quad t_n = t_{(n+1)-1} = \int_0^\infty s^n s d\nu(s) = \int_0^\infty s^n d\mu(s), \quad n \in \mathbb{Z}_+.$$

It is clear that  $\mu(\{0\}) = 0$  and thus

$$\begin{aligned} \int_0^\infty \frac{1}{s} d\mu(s) &= \int_{(0,\infty)} d\nu(s) = \nu((0,\infty)) \\ &= \int_{[0,\infty)} s^0 d\nu(s) - \nu(\{0\}) = \vartheta - \nu(\{0\}), \end{aligned}$$

<sup>1</sup> We adhere to the convention that  $\frac{1}{0} := \infty$ . Hence,  $\int_0^\infty \frac{1}{s} d\mu(s) < \infty$  implies  $\mu(\{0\}) = 0$ .

which implies that  $\int_0^\infty \frac{1}{s} d\mu(s) \leq \vartheta$ . This, combined with (2.4.4), shows that  $\mu \in \mathcal{M}_0(\vartheta)$ . Since  $\nu(\mathbb{R}_+) = \vartheta$ , we deduce from (2.4.2) and the definition of  $\mu$  that

$$\begin{aligned} \nu_\mu(\sigma) &= \int_{\sigma \setminus \{0\}} \frac{1}{s} d\mu(s) + \left( \vartheta - \int_0^\infty \frac{1}{s} d\mu(s) \right) \delta_0(\sigma \cap \{0\}) \\ &= \nu(\sigma \setminus \{0\}) + \left( \vartheta - \nu((0, \infty)) \right) \delta_0(\sigma \cap \{0\}) \\ &= \nu(\sigma \setminus \{0\}) + \nu(\{0\}) \delta_0(\sigma \cap \{0\}) = \nu(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+), \end{aligned}$$

which yields  $\nu_\mu = \nu$ .

We have proved that, under the assumption (i), the mapping  $\mathcal{M}_0(\vartheta) \ni \mu \rightarrow \nu_\mu \in \mathcal{M}_{-1}(\vartheta)$  is well-defined and surjective. Its injectivity follows from the equality

$$\mu(\sigma) = \mu(\sigma \setminus \{0\}) = \int_{\sigma \setminus \{0\}} s d\nu_\mu(s), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+), \mu \in \mathcal{M}_0(\vartheta).$$

This yields the determinacy part of the conclusion.  $\square$

**REMARK 2.4.2.** Suppose that  $\{t_n\}_{n=0}^\infty$  is a determinate Stieltjes moment sequence with a representing measure  $\mu$ . If  $\int_0^\infty \frac{1}{s} d\mu(s) = \infty$ , then  $(\vartheta, t_0, t_1, \dots)$  is never a Stieltjes moment sequence. In turn, if  $\int_0^\infty \frac{1}{s} d\mu(s) < \infty$ , then  $(\vartheta, t_0, t_1, \dots)$  is a determinate Stieltjes moment sequence if  $\vartheta \geq \int_0^\infty \frac{1}{s} d\mu(s)$ , and it is not a Stieltjes moment sequence if  $\vartheta < \int_0^\infty \frac{1}{s} d\mu(s)$ .

**REMARK 2.4.3.** Under the assumptions of Lemma 2.4.1, if  $\{t_{n-1}\}_{n=0}^\infty$  is a Stieltjes moment sequence and  $t_0 > 0$ , then  $t_n > 0$  for all  $n \in \mathbb{Z}_+$  and

$$\sup_{n \in \mathbb{Z}_+} \frac{t_n^2}{t_{2n+1}} \leq \int_0^\infty \frac{1}{s} d\mu(s) \leq \vartheta, \quad \mu \in \mathcal{M}_0(\vartheta).$$

Indeed, since  $t_0 > 0$  and  $\mu(\{0\}) = 0$ , we see that  $t_n > 0$  for all  $n \in \mathbb{Z}_+$ . Thus

$$t_n^2 = \left( \int_{(0, \infty)} s^{-1/2} s^{n+1/2} d\mu(s) \right)^2 \leq \int_0^\infty \frac{1}{s} d\mu(s) \int_0^\infty s^{2n+1} d\mu(s), \quad n \in \mathbb{Z}_+.$$

Note that if  $\{t_n\}_{n=0}^\infty$  is indeterminate, then there is a smallest  $\vartheta$  for which the sequence  $\{t_{n-1}\}_{n=0}^\infty$  is a Stieltjes moment sequence (see [18] for more details).

### 3. A General Setting for Subnormality

**3.1. Criteria for subnormality.** The only known general characterization of subnormality of unbounded Hilbert space operators is due to Bishop and Foias (cf. [5, 15]; see also [46] for a new approach via sesquilinear selection of elementary spectral measures). Since this characterization refers to semispectral measures (or elementary spectral measures), it seems to be useless in the context of weighted shifts on directed trees. The other known criteria for subnormality require the operator in question to have an invariant domain (with the exception of [50]). Since a closed subnormal operator with an invariant domain is automatically bounded (cf. [29, Theorem 3.3]) and a weighted shift operator  $S_\lambda$  on a directed tree is always closed (cf. [17, Proposition 3.1.2]), we have to find a smaller subspace of  $\mathcal{D}(S_\lambda)$  which is an invariant core of  $S_\lambda$ . This will enable us to apply the aforesaid criteria for subnormality of operators with invariant domains in the context of weighted shift operators on directed trees.

Using a recent result from [7], we obtain the following criterion for subnormality which is a key tool for proving Theorem 5.1.1.



**THEOREM 3.1.1.** *Let  $\{S_\omega\}_{\omega \in \Omega}$  be a net of subnormal operators in a complex Hilbert space  $\mathcal{H}$  and let  $S$  be a densely defined operator in  $\mathcal{H}$ . Suppose that there is a subset  $\mathcal{X}$  of  $\mathcal{H}$  such that*

- (i)  $\mathcal{X} \subseteq \mathcal{D}^\infty(S) \cap \bigcap_{\omega \in \Omega} \mathcal{D}^\infty(S_\omega)$ ,
- (ii)  $\mathcal{F} := \text{LIN} \bigcup_{n=0}^\infty S^n(\mathcal{X})$  is a core of  $S$ ,
- (iii)  $\langle S^m x, S^n y \rangle = \lim_{\omega \in \Omega} \langle S_\omega^m x, S_\omega^n y \rangle$  for all  $x, y \in \mathcal{X}$  and  $m, n \in \mathbb{Z}_+$ .

*Then  $S$  is subnormal.*

**PROOF.** Set  $\mathcal{F}_\omega = \text{LIN} \bigcup_{n=0}^\infty S_\omega^n(\mathcal{X})$  for  $\omega \in \Omega$ . It is clear that  $S_\omega|_{\mathcal{F}_\omega}$  is a subnormal operator in  $\overline{\mathcal{F}_\omega}$  with an invariant domain.

Take a finite system  $\{a_{p,q}^{i,j}\}_{p,q=0,\dots,n}^{i,j=1,\dots,m}$  of complex numbers such that

$$\sum_{i,j=1}^m \sum_{p,q=0}^n a_{p,q}^{i,j} \lambda^p \bar{\lambda}^q z_i \bar{z}_j \geq 0, \quad \lambda, z_1, \dots, z_m \in \mathbb{C}.$$

Let  $f_1, \dots, f_m$  be arbitrary vectors in  $\mathcal{F}$ . Then for every  $i \in \{1, \dots, m\}$ , there exists a positive integer  $r$  and a system  $\{\zeta_{x,k}^{(i)} : x \in \mathcal{X}, k = 1, \dots, r\}$  of complex numbers such that the set  $\{x \in \mathcal{X} : \zeta_{x,k}^{(i)} \neq 0\}$  is finite for every  $k \in \{1, \dots, r\}$ , and  $f_i = \sum_{x \in \mathcal{X}} \sum_{k=1}^r \zeta_{x,k}^{(i)} S^k x$ . Set  $f_{i,\omega} = \sum_{x \in \mathcal{X}} \sum_{k=1}^r \zeta_{x,k}^{(i)} S_\omega^k x$  for  $i \in \{1, \dots, m\}$  and  $\omega \in \Omega$ . Then  $f_{i,\omega} \in \mathcal{F}_\omega$  for all  $i \in \{1, \dots, m\}$  and  $\omega \in \Omega$ . Applying [7, Theorem 21] to the subnormal operators  $S_\omega|_{\mathcal{F}_\omega}$ , we get

$$\begin{aligned} \sum_{i,j=1}^m \sum_{p,q=0}^n a_{p,q}^{i,j} \langle S^p f_i, S^q f_j \rangle &= \sum_{i,j=1}^m \sum_{p,q=0}^n \sum_{x,y \in \mathcal{X}} \sum_{k,l=1}^r a_{p,q}^{i,j} \zeta_{x,k}^{(i)} \overline{\zeta_{y,l}^{(j)}} \langle S^{p+k} x, S^{q+l} y \rangle \\ &\stackrel{\text{(iii)}}{=} \lim_{\omega \in \Omega} \sum_{i,j=1}^m \sum_{p,q=0}^n \sum_{x,y \in \mathcal{X}} \sum_{k,l=1}^r a_{p,q}^{i,j} \zeta_{x,k}^{(i)} \overline{\zeta_{y,l}^{(j)}} \langle S_\omega^{p+k} x, S_\omega^{q+l} y \rangle \\ &= \lim_{\omega \in \Omega} \sum_{i,j=1}^m \sum_{p,q=0}^n a_{p,q}^{i,j} \langle S_\omega^p f_{i,\omega}, S_\omega^q f_{j,\omega} \rangle \geq 0. \end{aligned}$$

This means that the operator  $S|_{\mathcal{F}}$  satisfies condition (ii) of [7, Theorem 21]. Since  $S|_{\mathcal{F}}$  has an invariant domain, we deduce from [7, Theorem 21] that  $S|_{\mathcal{F}}$  is subnormal. Combining the latter with the assumption that  $\mathcal{F}$  is a core of  $S$ , we see that  $S$  itself is subnormal. This completes the proof.  $\square$

We say that a densely defined operator  $S$  in a complex Hilbert space  $\mathcal{H}$  is *cyclic* with a *cyclic vector*  $e \in \mathcal{H}$  if  $e \in \mathcal{D}^\infty(S)$  and  $\text{LIN}\{S^n e : n = 0, 1, \dots\}$  is a core of  $S$ .

**COROLLARY 3.1.2.** *Let  $\{S_\omega\}_{\omega \in \Omega}$  be a net of subnormal operators in a complex Hilbert space  $\mathcal{H}$  and let  $S$  be a cyclic operator in  $\mathcal{H}$  with a cyclic vector  $e$  such that*

- (i)  $e \in \bigcap_{\omega \in \Omega} \mathcal{D}^\infty(S_\omega)$ ,
- (ii)  $\langle S^m e, S^n e \rangle = \lim_{\omega \in \Omega} \langle S_\omega^m e, S_\omega^n e \rangle$  for all  $m, n \in \mathbb{Z}_+$ .

*Then  $S$  is subnormal.*

**3.2. Necessity.** Let us recall a well-known fact that  $C^\infty$ -vectors of a subnormal operator always generate Stieltjes moment sequences.

**PROPOSITION 3.2.1.** *If  $S$  is a subnormal operator in a complex Hilbert space  $\mathcal{H}$ , then  $\mathcal{D}^\infty(S) = \mathcal{S}(S)$ , where  $\mathcal{S}(S)$  stands for the set of all vectors  $f \in \mathcal{D}^\infty(S)$  such that the sequence  $\{\|S^n f\|^2\}_{n=0}^\infty$  is a Stieltjes moment sequence.*

PROOF. Let  $N$  be a normal extension of  $S$  acting in a complex Hilbert space  $\mathcal{K} \supseteq \mathcal{H}$  and let  $E$  be the spectral measure of  $N$ . Define the mapping  $\phi: \mathbb{C} \rightarrow \mathbb{R}_+$  by  $\phi(z) = |z|^2$ ,  $z \in \mathbb{C}$ . Since evidently  $\mathcal{D}^\infty(S) \subseteq \mathcal{D}^\infty(N)$ , we deduce from the measure transport theorem (cf. [4, Theorem 5.4.10]) that for every  $f \in \mathcal{D}^\infty(S)$ ,

$$\begin{aligned} \|S^n f\|^2 &= \|N^n f\|^2 = \left\| \int_{\mathbb{C}} z^n E(dz) f \right\|^2 \\ &= \int_{\mathbb{C}} \phi(z)^n \langle E(dz) f, f \rangle = \int_0^\infty t^n \langle F(dt) f, f \rangle, \quad n \in \mathbb{Z}_+, \end{aligned}$$

where  $F$  is the spectral measure on  $\mathbb{R}_+$  given by  $F(\sigma) = E(\phi^{-1}(\sigma))$  for  $\sigma \in \mathfrak{B}(\mathbb{R}_+)$ . This implies that  $\mathcal{D}^\infty(S) \subseteq \mathcal{S}(S)$ .  $\square$

It follows from Proposition 3.2.1 that if  $S$  is a subnormal operator with invariant domain, then  $S$  is densely defined and  $\mathcal{D}(S) = \mathcal{S}(S)$ . One might expect that the reverse implication holds as well. This is really the case for bounded operators (cf. [25]) and for some unbounded operators that have sufficiently many analytic vectors (cf. [42, Theorem 7]). In Section 5.3 we show that this is also the case for weighted shifts on directed trees that have sufficiently many quasi-analytic vectors (see Theorem 5.3.1). However, in general, this is not the case. Indeed, one can construct a densely defined operator  $N$  in a complex Hilbert space  $\mathcal{H}$  which is not subnormal and which has the following properties (see [8, 34, 39]):

$$(3.2.1) \quad N(\mathcal{D}(N)) \subseteq \mathcal{D}(N), \quad \mathcal{D}(N) \subseteq \mathcal{D}(N^*), \quad N^*(\mathcal{D}(N)) \subseteq \mathcal{D}(N)$$

$$(3.2.2) \quad \text{and } N^* N f = N N^* f \text{ for all } f \in \mathcal{D}(N).$$

We show that for such  $N$ ,  $\mathcal{D}(N) = \mathcal{S}(N)$ . Indeed, by (3.2.1) and (3.2.2), we have

$$\sum_{k,l=0}^n \|N^{k+l} f\|^2 \alpha_k \overline{\alpha_l} = \sum_{k,l=0}^n \langle (N^* N)^{k+l} f, f \rangle \alpha_k \overline{\alpha_l} = \left\| \sum_{k=0}^n \alpha_k (N^* N)^k f \right\|^2 \geq 0,$$

for all  $f \in \mathcal{D}(N)$ ,  $n \in \mathbb{Z}_+$  and  $\alpha_0, \dots, \alpha_n \in \mathbb{C}$ , which means that the sequence  $\{\|N^n f\|^2\}_{n=0}^\infty$  is positive definite for every  $f \in \mathcal{D}(N)$ . Replacing  $f$  by  $Nf$ , we see that the sequence  $\{\|N^{n+1} f\|^2\}_{n=0}^\infty$  is positive definite for every  $f \in \mathcal{D}(N)$ . Applying the Stieltjes theorem, we conclude that  $\mathcal{D}(N) = \mathcal{S}(N)$ .

## 4. Towards Subnormality of Weighted Shifts

**4.1. A consistency condition.** Applying Proposition 3.2.1, we get.

PROPOSITION 4.1.1. *Let  $S_\lambda$  be a weighted shift on a directed tree  $\mathcal{T}$  with weights  $\lambda = \{\lambda_v\}_{v \in V^\circ}$  such that  $\mathcal{E}_V \subseteq \mathcal{D}^\infty(S_\lambda)$ . If  $S_\lambda$  is subnormal, then for every  $u \in V$  the sequence  $\{\|S_\lambda^n e_u\|^2\}_{n=0}^\infty$  is a Stieltjes moment sequence.*

The converse of the implication in Proposition 4.1.1 is valid for bounded weighted shifts on directed trees (the unbounded case is discussed in Theorem 5.3.1).

THEOREM 4.1.2 ([17, Theorem 6.1.3]). *Let  $S_\lambda \in \mathcal{B}(\ell^2(V))$  be a weighted shift on a directed tree  $\mathcal{T}$  with weights  $\lambda = \{\lambda_v\}_{v \in V^\circ}$ . Then  $S_\lambda$  is subnormal if and only if  $\{\|S_\lambda^n e_u\|^2\}_{n=0}^\infty$  is a Stieltjes moment sequence for every  $u \in V$ .*

If  $S_\lambda$  is a subnormal weighted shift on a directed tree  $\mathcal{T}$ , then in view of Proposition 4.1.1 we can attach to each vertex  $u \in V$  a representing measure  $\mu_u$  of the Stieltjes moment sequence  $\{\|S_\lambda^n e_u\|^2\}_{n=0}^\infty$  (of course, since the sequence  $\{\|S_\lambda^n e_u\|^2\}_{n=0}^\infty$  is not determinate in general, we have to choose one of them); note

that any such  $\mu_u$  is a probability measure. Hence, it is tempting to find relationships between these representing measures. This has been done in the case of bounded weighted shifts in [17, Lemma 6.1.10]. What is stated below is an adaptation of this lemma (and its proof) to the unbounded case. As opposed to the bounded case, implication  $1^\circ \Rightarrow 2^\circ$  of Lemma 4.1.3 below is not true in general (cf. [18]).

LEMMA 4.1.3. *Let  $S_\lambda$  be a weighted shift on a directed tree  $\mathcal{T}$  with weights  $\lambda = \{\lambda_v\}_{v \in V^\circ}$  such that  $\mathcal{E}_V \subseteq \mathcal{D}^\infty(S_\lambda)$ . Let  $u \in V'$ . Suppose that for every  $v \in \text{Chi}(u)$  the sequence  $\{\|S_\lambda^n e_v\|^2\}_{n=0}^\infty$  is a Stieltjes moment sequence with a representing measure  $\mu_v$ . Consider the following two conditions<sup>2</sup>:*

- $1^\circ$   $\{\|S_\lambda^n e_u\|^2\}_{n=0}^\infty$  is a Stieltjes moment sequence,
- $2^\circ$   $S_\lambda$  satisfies the consistency condition at the vertex  $u$ , i.e.,

$$(4.1.1) \quad \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 \int_0^\infty \frac{1}{s} d\mu_v(s) \leq 1.$$

Then the following assertions are valid:

- (i) if  $2^\circ$  holds, then so does  $1^\circ$  and the positive Borel measure  $\mu_u$  on  $\mathbb{R}_+$  defined by

$$(4.1.2) \quad \mu_u(\sigma) = \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 \int_\sigma^\infty \frac{1}{s} d\mu_v(s) + \varepsilon_u \delta_0(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),$$

with

$$(4.1.3) \quad \varepsilon_u = 1 - \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 \int_0^\infty \frac{1}{s} d\mu_v(s)$$

is a representing measure of  $\{\|S_\lambda^n e_u\|^2\}_{n=0}^\infty$ ,

- (ii) if  $1^\circ$  holds and  $\{\|S_\lambda^{n+1} e_u\|^2\}_{n=0}^\infty$  is determinate, then  $2^\circ$  holds, the Stieltjes moment sequence  $\{\|S_\lambda^n e_u\|^2\}_{n=0}^\infty$  is determinate and its unique representing measure  $\mu_u$  is given by (4.1.2) and (4.1.3).

PROOF. Define the positive Borel measure  $\mu$  on  $\mathbb{R}_+$  by

$$\mu(\sigma) = \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 \mu_v(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+).$$

It is a matter of routine to show that

$$(4.1.4) \quad \int_0^\infty f d\mu = \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 \int_0^\infty f d\mu_v$$

for every Borel function  $f: [0, \infty) \rightarrow [0, \infty]$ . Using the inclusion  $\mathcal{E}_V \subseteq \mathcal{D}^\infty(S_\lambda)$  and applying Lemma 2.3.1 (iii) twice, we obtain

$$\begin{aligned} \|S_\lambda^{n+1} e_u\|^2 &= \sum_{w \in \text{Chi}^{(n+1)}(u)} |\lambda_{u|w}|^2 \\ &\stackrel{(2.2.5)}{=} \sum_{v \in \text{Chi}(u)} \sum_{w \in \text{Chi}^{(n)}(v)} |\lambda_{u|w}|^2 \\ &\stackrel{(2.3.4)}{=} \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 \sum_{w \in \text{Chi}^{(n)}(v)} |\lambda_{v|w}|^2 \end{aligned}$$

<sup>2</sup> We adhere to the standard convention that  $0 \cdot \infty = 0$ ; see also footnote 1.

$$= \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 \|S_{\lambda}^n e_v\|^2, \quad n \in \mathbb{Z}_+.$$

This implies that

$$\|S_{\lambda}^{n+1} e_u\|^2 = \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 \int_0^\infty s^n d\mu_v(s) \stackrel{(4.1.4)}{=} \int_0^\infty s^n d\mu(s), \quad n \in \mathbb{Z}_+.$$

Hence the sequence  $\{\|S_{\lambda}^{n+1} e_u\|^2\}_{n=0}^\infty$  is a Stieltjes moment sequence with a representing measure  $\mu$ .

Set  $t_n = \|S_{\lambda}^{n+1} e_u\|^2$  for  $n \in \mathbb{Z}_+$ , and  $t_{-1} = 1$ . Note that

$$t_{n-1} = \|S_{\lambda}^n e_u\|^2, \quad n \in \mathbb{Z}_+.$$

Suppose that  $2^\circ$  holds. Then, by (4.1.1) and (4.1.4), we have  $\int_0^\infty \frac{1}{s} d\mu(s) \leq 1$ . Applying implication (iii) $\Rightarrow$ (i) of Lemma 2.4.1, we see that  $1^\circ$  holds, and, by (4.1.4), the measure  $\mu_u$  defined by (4.1.2) and (4.1.3) is a representing measure of the Stieltjes moment sequence  $\{\|S_{\lambda}^n e_u\|^2\}_{n=0}^\infty$ .

Suppose now that  $1^\circ$  holds and the Stieltjes moment sequence  $\{\|S_{\lambda}^{n+1} e_u\|^2\}_{n=0}^\infty$  is determinate. It follows from implication (i) $\Rightarrow$ (iii) of Lemma 2.4.1 that there is a representing measure  $\mu'$  of  $\{\|S_{\lambda}^{n+1} e_u\|^2\}_{n=0}^\infty$  such that  $\int_0^\infty \frac{1}{s} d\mu'(s) \leq 1$ . Since  $\{\|S_{\lambda}^{n+1} e_u\|^2\}_{n=0}^\infty$  is determinate, we get  $\mu' = \mu$ , which implies  $2^\circ$ . The remaining part of assertion (ii) follows from the last assertion of Lemma 2.4.1.  $\square$

Now we prove that the determinacy of appropriate Stieltjes moment sequences attached to a weighted shift on a directed tree implies the existence of a consistent system of measures (see also Theorem 5.1.3). As shown in [18], Lemma 4.1.4 below is no longer true if the assumption on determinacy is dropped (by Lemma 4.2.2 (iv), the converse of Lemma 4.1.4 is true without assuming determinacy).

**LEMMA 4.1.4.** *Let  $S_{\lambda}$  be a weighted shift on a directed tree  $\mathcal{T}$  with weights  $\lambda = \{\lambda_v\}_{v \in V^\circ}$  such that  $\mathcal{E}_V \subseteq \mathcal{D}^\infty(S_{\lambda})$ . Assume that for every  $u \in V'$ , the sequence  $\{\|S_{\lambda}^n e_u\|^2\}_{n=0}^\infty$  is a Stieltjes moment sequence, and that the Stieltjes moment sequence  $\{\|S_{\lambda}^{n+1} e_u\|^2\}_{n=0}^\infty$  (cf. (2.4.1)) is determinate. Then there exist a system  $\{\mu_u\}_{u \in V}$  of Borel probability measures on  $\mathbb{R}_+$  and a system  $\{\varepsilon_u\}_{u \in V}$  of nonnegative real numbers that satisfy (4.1.2) for every  $u \in V$ .*

**PROOF.** By Lemma 2.4.1, the Stieltjes moment sequence  $\{\|S_{\lambda}^n e_u\|^2\}_{n=0}^\infty$  is determinate for every  $u \in V'$ . For  $u \in V'$ , we denote by  $\mu_u$  the unique representing measure of  $\{\|S_{\lambda}^n e_u\|^2\}_{n=0}^\infty$ . If  $u \in V \setminus V'$ , then we put  $\mu_u = \delta_0$ . Using Lemma 4.1.3 (ii), we verify that the system  $\{\mu_u\}_{u \in V}$  satisfies (4.1.2) with  $\{\varepsilon_u\}_{u \in V}$  defined by (4.1.3). This completes the proof.  $\square$

**4.2. Consistent systems of measures.** In this section we prove some important properties of consistent systems of Borel probability measures on  $\mathbb{R}_+$  attached to a directed tree. They will be used in the proof of Theorem 5.1.1.

**LEMMA 4.2.1.** *Let  $\mathcal{T}$  be a directed tree. Suppose that  $\{\lambda_v\}_{v \in V^\circ}$  is a system of complex numbers,  $\{\varepsilon_v\}_{v \in V}$  is a system of nonnegative real numbers and  $\{\mu_v\}_{v \in V}$  is a system of Borel probability measures on  $\mathbb{R}_+$  satisfying (4.1.2) for every  $u \in V$ . Then the following assertions hold:*

(i) for every  $u \in V$ ,  $\sum_{v \in \text{Chi}(u)} |\lambda_v|^2 \int_0^\infty \frac{1}{s} d\mu_v(s) \leq 1$  and

$$\varepsilon_u = 1 - \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 \int_0^\infty \frac{1}{s} d\mu_v(s),$$

(ii) for every  $u \in V$ ,  $\mu_u(\{0\}) = 0$  if and only if  $\varepsilon_u = 0$ ,

(iii) for every  $v \in V^\circ$ , if  $\lambda_v \neq 0$ , then  $\mu_v(\{0\}) = 0$ ,

(iv) for every  $u \in V$ ,

$$(4.2.1) \quad \mu_u(\sigma) = \sum_{v \in \text{Chi}^{(n)}(u)} |\lambda_{u|v}|^2 \int_\sigma \frac{1}{s^n} d\mu_v(s) + \varepsilon_u \delta_0(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+), n \geq 1.$$

PROOF. (i) Substitute  $\sigma = \mathbb{R}_+$  into (4.1.2) and note that  $\mu_u(\mathbb{R}_+) = 1$ .

(ii) & (iii) Substitute  $\sigma = \{0\}$  into (4.1.2).

(iv) We use induction on  $n$ . The case of  $n = 1$  coincides with (4.1.2). Suppose that (4.2.1) is valid for a fixed integer  $n \geq 1$ . Then combining (4.1.2) with (4.2.1), we see that

$$(4.2.2) \quad \begin{aligned} \mu_u(\sigma) &= \sum_{v \in \text{Chi}^{(n)}(u)} |\lambda_{u|v}|^2 \sum_{w \in \text{Chi}(v)} |\lambda_w|^2 \int_\sigma \frac{1}{s^{n+1}} d\mu_w(s) \\ &\quad + \sum_{v \in \text{Chi}^{(n)}(u)} |\lambda_{u|v}|^2 \int_\sigma \frac{1}{s^n} d(\varepsilon_v \delta_0)(s) + \varepsilon_u \delta_0(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+). \end{aligned}$$

Since  $\mu_u$  is a finite positive measure and  $n \geq 1$ , we deduce from (4.2.2) that  $\varepsilon_v = 0$  whenever  $\lambda_{u|v} \neq 0$ , and thus

$$(4.2.3) \quad \sum_{v \in \text{Chi}^{(n)}(u)} |\lambda_{u|v}|^2 \int_\sigma \frac{1}{s^n} d(\varepsilon_v \delta_0)(s) = 0.$$

It follows from (4.2.2) and (4.2.3) that

$$\begin{aligned} \mu_u(\sigma) &= \sum_{v \in \text{Chi}^{(n)}(u)} \sum_{w \in \text{Chi}(v)} |\lambda_{u|v} \lambda_w|^2 \int_\sigma \frac{1}{s^{n+1}} d\mu_w(s) + \varepsilon_u \delta_0(\sigma) \\ &\stackrel{(2.2.6) \& (2.3.3)}{=} \sum_{w \in \text{Chi}^{(n+1)}(u)} |\lambda_{u|w}|^2 \int_\sigma \frac{1}{s^{n+1}} d\mu_w(s) + \varepsilon_u \delta_0(\sigma). \end{aligned}$$

This completes the proof.  $\square$

LEMMA 4.2.2. Let  $\mathcal{T}$  be a directed tree. Suppose that  $\lambda = \{\lambda_v\}_{v \in V^\circ}$  is a system of complex numbers,  $\{\varepsilon_v\}_{v \in V}$  is a system of nonnegative real numbers and  $\{\mu_v\}_{v \in V}$  is a system of Borel probability measures on  $\mathbb{R}_+$  satisfying (4.1.2) for every  $u \in V$ . Let  $S_\lambda$  be a weighted shift on the directed tree  $\mathcal{T}$  with weights  $\lambda$ . Then the following assertions hold:

(i) for all  $u \in V$  and  $n \in \mathbb{N}$ ,

$$(4.2.4) \quad \int_0^\infty s^n d\mu_u(s) = \sum_{v \in \text{Chi}^{(n)}(u)} |\lambda_{u|v}|^2,$$

(ii) if  $\text{Chi}^{(n)}(u) = \emptyset$  for some  $u \in V$  and  $n \in \mathbb{N}$ , then  $\mu_v = \delta_0$  for all  $v \in \text{Des}(u)$ ,

(iii)  $\mathcal{E}_V \subseteq \mathcal{D}^\infty(S_\lambda)$  if and only if  $\int_0^\infty s^n d\mu_u(s) < \infty$  for all  $n \in \mathbb{Z}_+$  and  $u \in V$ ,

(iv) if  $\mathcal{E}_V \subseteq \mathcal{D}^\infty(S_\lambda)$ , then for all  $u \in V$  and  $n \in \mathbb{Z}_+$ ,

$$(4.2.5) \quad \|S_\lambda^n e_u\|^2 = \int_0^\infty s^n d\mu_u(s),$$

(v)  $S_\lambda \in \mathcal{B}(\ell^2(V))$  if and only if there exists a real number  $M \geq 0$  such that  $\text{supp } \mu_u \subseteq [0, M]$  for every  $u \in V$ .

PROOF. (i) Substituting  $\sigma = \{0\}$  into (4.2.1), we see that for every  $v \in \text{Chi}^{(n)}(u)$ , either  $\lambda_{u|v} = 0$ , or  $\lambda_{u|v} \neq 0$  and  $\mu_v(\{0\}) = 0$ . This and (4.2.1) lead to (4.2.4).

(ii) It follows from (4.2.4) that  $\int_0^\infty s^n d\mu_u(s) = 0$  (recall the convention that  $\sum_{v \in \emptyset} x_v = 0$ ). This and  $n \geq 1$  implies that  $\mu_u((0, \infty)) = 0$ . Since  $\mu_u(\mathbb{R}_+) = 1$ , we deduce that  $\mu_u = \delta_0$ .

If  $v \in \text{Des}(u) \setminus \{u\}$ , then by (2.2.7) there exists  $k \in \mathbb{N}$  such that  $v \in \text{Chi}^{(k)}(u)$ . Since  $\text{Chi}(\cdot)$  is a monotonically increasing set-function, we infer from (2.2.3) that  $\text{Chi}^{(n)}(v) \subseteq \text{Chi}^{(n+k)}(u) = \emptyset$ . By the previous argument applied to  $v$  in place of  $u$ , we get  $\mu_v = \delta_0$ .

Assertions (iii) and (iv) follow from (i) and Lemma 2.3.1.

(v) To prove the “only if” part, note that

$$\lim_{n \rightarrow \infty} \left( \int_0^\infty s^n d\mu_u(s) \right)^{1/n} \stackrel{(4.2.5)}{=} \lim_{n \rightarrow \infty} (\|S_\lambda^n e_u\|^{1/n})^2 \leq \|S_\lambda\|^2,$$

which implies that  $\text{supp } \mu_u \subseteq [0, \|S_\lambda\|^2]$  (cf. [31, page 71]). The proof of the converse implication goes as follows. In view of (4.2.4), we have

$$\sum_{v \in \text{Chi}(u)} |\lambda_v|^2 = \int_0^\infty s d\mu_u(s) \leq M, \quad u \in V,$$

which, by [17, Proposition 3.1.8], implies that  $S_\lambda \in \mathcal{B}(\ell^2(V))$  and  $\|S_\lambda\| \leq \sqrt{M}$ .  $\square$

## 5. Criteria for Subnormality of Weighted Shifts

**5.1. Arbitrary weights.** After all these preparations we can prove the main criterion for subnormality of unbounded weighted shifts on directed trees. It is written in terms of consistent systems of measures.

**THEOREM 5.1.1.** *Let  $S_\lambda$  be a weighted shift on a directed tree  $\mathcal{T}$  with weights  $\lambda = \{\lambda_v\}_{v \in V^\circ}$  such that  $\mathcal{E}_V \subseteq \mathcal{D}^\infty(S_\lambda)$ . Suppose that there exist a system  $\{\mu_v\}_{v \in V}$  of Borel probability measures on  $\mathbb{R}_+$  and a system  $\{\varepsilon_v\}_{v \in V}$  of nonnegative real numbers that satisfy (4.1.2) for every  $u \in V$ . Then  $S_\lambda$  is subnormal.*

PROOF. For a fixed positive integer  $i$ , we define the system  $\lambda^{(i)} = \{\lambda_v^{(i)}\}_{v \in V^\circ}$  of complex numbers, the system  $\{\mu_v^{(i)}\}_{v \in V}$  of Borel probability measures on  $\mathbb{R}_+$  and the system  $\{\varepsilon_v^{(i)}\}_{v \in V}$  of nonnegative real numbers by

$$(5.1.1) \quad \lambda_v^{(i)} = \begin{cases} \lambda_v \sqrt{\frac{\mu_v([0, i])}{\mu_{\text{par}(v)}([0, i])}} & \text{if } \mu_{\text{par}(v)}([0, i]) > 0, \\ 0 & \text{if } \mu_{\text{par}(v)}([0, i]) = 0, \end{cases} \quad v \in V^\circ,$$

$$(5.1.2) \quad \mu_v^{(i)}(\sigma) = \begin{cases} \frac{\mu_v(\sigma \cap [0, i])}{\mu_v([0, i])} & \text{if } \mu_v([0, i]) > 0, \\ \delta_0(\sigma) & \text{if } \mu_v([0, i]) = 0, \end{cases} \quad \sigma \in \mathfrak{B}(\mathbb{R}_+), v \in V,$$

$$(5.1.3) \quad \varepsilon_v^{(i)} = \begin{cases} \frac{\varepsilon_v}{\mu_v([0, i])} & \text{if } \mu_v([0, i]) > 0, \\ 1 & \text{if } \mu_v([0, i]) = 0, \end{cases} \quad v \in V.$$

Our first goal is to show that the following equality holds for all  $u \in V$  and  $i \in \mathbb{N}$ ,

$$(5.1.4) \quad \mu_u^{(i)}(\sigma) = \sum_{v \in \text{Chi}(u)} |\lambda_v^{(i)}|^2 \int_{\sigma} \frac{1}{s} d\mu_v^{(i)}(s) + \varepsilon_u^{(i)} \delta_0(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+).$$

For this fix  $u \in V$  and  $i \in \mathbb{N}$ . If  $\mu_u([0, i]) = 0$ , then, according to our definitions, we have  $\lambda_v^{(i)} = 0$  for all  $v \in \text{Chi}(u)$ ,  $\mu_u^{(i)} = \delta_0$  and  $\varepsilon_u^{(i)} = 1$ , which means that the equality (5.1.4) holds. Consider now the case of  $\mu_u([0, i]) > 0$ . It follows from (4.1.2) that

$$(5.1.5) \quad \mu_u(\sigma \cap [0, i]) = \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 \int_{\sigma \cap [0, i]} \frac{1}{s} d\mu_v(s) + \varepsilon_u \delta_0(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+).$$

If  $v \in \text{Chi}(u)$  (equivalently:  $u = \text{par}(v)$ ), then by (5.1.1) and (5.1.2) we have

$$(5.1.6) \quad \frac{|\lambda_v|^2}{\mu_u([0, i])} \int_{\sigma \cap [0, i]} \frac{1}{s} d\mu_v(s) = \begin{cases} |\lambda_v^{(i)}|^2 \int_{\sigma} \frac{1}{s} d\mu_v^{(i)}(s) & \text{if } \mu_v([0, i]) > 0, \\ 0 & \text{if } \mu_v([0, i]) = 0, \end{cases} \\ = |\lambda_v^{(i)}|^2 \int_{\sigma} \frac{1}{s} d\mu_v^{(i)}(s),$$

where the last equality holds because  $\lambda_v^{(i)} = 0$  whenever  $\mu_v([0, i]) = 0$ . Dividing both sides of (5.1.5) by  $\mu_u([0, i])$  and using (5.1.6), we obtain (5.1.4).

Let  $S_{\lambda^{(i)}}$  be the weighted shift on  $\mathcal{T}$  with weights  $\lambda^{(i)}$ . Since, by (5.1.2),  $\text{supp } \mu_u^{(i)} \subseteq [0, i]$  for every  $u \in V$ , we infer from (5.1.4) and Lemma 4.2.2 (v), applied to the triplet  $(\lambda^{(i)}, \{\mu_v^{(i)}\}_{v \in V}, \{\varepsilon_v^{(i)}\}_{v \in V})$ , that  $S_{\lambda^{(i)}} \in \mathbf{B}(\ell^2(V))$ . In turn, (5.1.4) and Lemma 4.2.2 (iv) (applied to the same triplet) imply that for every  $u \in V$ ,  $\{\|S_{\lambda^{(i)}}^n e_u\|^2\}_{n=0}^\infty$  is a Stieltjes moment sequence (with a representing measure  $\mu_u^{(i)}$ ). Hence, by Theorem 4.1.2, the operator  $S_{\lambda^{(i)}}$  is subnormal.

Since  $\mu_u$ ,  $u \in V$ , are Borel probability measures on  $\mathbb{R}_+$ , we have

$$(5.1.7) \quad \lim_{i \rightarrow \infty} \mu_u([0, i]) = 1, \quad u \in V.$$

Hence, for every  $u \in V$  there exists a positive integer  $\kappa_u$  such that

$$(5.1.8) \quad \mu_u([0, i]) > 0, \quad i \in \mathbb{N}, i \geq \kappa_u.$$

Note that

$$(5.1.9) \quad \lim_{i \rightarrow \infty} \lambda_v^{(i)} = \lambda_v, \quad v \in V^\circ.$$

Indeed, if  $v \in V^\circ$ , then (5.1.1) and (5.1.8) yield  $\lambda_v^{(i)} = \lambda_v \sqrt{\frac{\mu_v([0, i])}{\mu_{\text{par}(v)}([0, i])}}$  for all integers  $i \geq \kappa_{\text{par}(v)}$ . This, combined with (5.1.7), gives (5.1.9). By (5.1.2), (5.1.8), (5.1.4)

and Lemma 4.2.2 (iv), applied to  $S_{\lambda^{(i)}}$ , we have

$$\|S_{\lambda^{(i)}}^n e_u\|^2 = \int_0^\infty s^n d\mu_u^{(i)}(s) = \frac{1}{\mu_u([0, i])} \int_{[0, i]} s^n d\mu_u(s), \quad n \in \mathbb{Z}_+, i \geq \kappa_u, u \in V.$$

This, together with (5.1.7) and Lemma 4.2.2 (iv), now applied to  $S_\lambda$ , implies that

$$(5.1.10) \quad \lim_{i \rightarrow \infty} \|S_{\lambda^{(i)}}^n e_u\|^2 = \int_0^\infty s^n d\mu_u(s) = \|S_\lambda^n e_u\|^2, \quad n \in \mathbb{Z}_+, u \in V.$$

It follows from (5.1.9), (5.1.10) and Proposition 2.3.2 that (2.3.8) holds. According to [17, Proposition 3.1.3 (vi)],  $\mathcal{E}_V$  is a core of  $S_\lambda$ . Hence  $\text{LIN} \bigcup_{n=0}^\infty S_\lambda^n(\mathcal{E}_V)$  is a core of  $S_\lambda$  as well. Applying (2.3.8) and Theorem 3.1.1 to the operators  $\{S_{\lambda^{(i)}}\}_{i=1}^\infty$  and  $S_\lambda$  with  $\mathcal{X} := \{e_u : u \in V\}$  completes the proof of Theorem 5.1.1.  $\square$

REMARK 5.1.2. In the proof of Theorem 5.1.1 we have used Proposition 2.3.2 which provides a general criterion for the validity of the approximation procedure (2.3.8). However, if the approximating triplets  $(\lambda^{(i)}, \{\mu_v^{(i)}\}_{v \in V}, \{\varepsilon_v^{(i)}\}_{v \in V})$ ,  $i = 1, 2, 3, \dots$ , are defined as in (5.1.1), (5.1.2) and (5.1.3), then

$$(5.1.11) \quad \lim_{i \rightarrow \infty} S_{\lambda^{(i)}}^n e_u = S_\lambda^n e_u, \quad u \in V, n \in \mathbb{Z}_+.$$

To prove this, we first show that for all  $u \in V$  and  $i \geq \kappa_u$  (see (5.1.8)),

$$(5.1.12) \quad \lambda_{u|u'}^{(i)} = \lambda_{u|u'} \sqrt{\frac{\mu_{u'}([0, i])}{\mu_u([0, i])}}, \quad u' \in \text{Chi}^{(n)}(u), n \in \mathbb{Z}_+.$$

Indeed, if  $n = 0$ , then (5.1.12) holds. Suppose that  $n \geq 1$ . If  $\mu_{\text{par}(u')}([0, i]) = 0$ , then  $n \geq 2$  and, by (5.1.1),  $\lambda_{u'}^{(i)} = 0$ , which implies that  $\lambda_{u|u'}^{(i)} = 0$ . Since  $\mu_{\text{par}(u')}([0, i]) = 0$ , we deduce from (4.1.2) (applied to  $u = \text{par}(u')$ ) that either  $\lambda_{u'} = 0$ , or  $\mu_{u'}([0, i]) = 0$ . In both cases, the right-hand side of (5.1.12) vanishes, and so (5.1.12) holds. In turn, if  $\mu_{\text{par}(u')}([0, i]) > 0$ , then we can define

$$j_0 = \min \left\{ j \in \{1, \dots, n\} : \mu_{\text{par}^k(u')}([0, i]) > 0 \text{ for all } k = 1, \dots, j \right\}.$$

Clearly,  $1 \leq j_0 \leq n$ . First, we consider the case where  $j_0 < n$ . Since, by (5.1.8),  $\mu_u([0, i]) > 0$ , we must have  $j_0 \leq n - 2$ . Thus  $\mu_{\text{par}^{j_0+1}(u')}([0, i]) = 0$ , which together with (2.3.2) and (5.1.1) implies that the left-hand side of (5.1.12) vanishes. Since  $\mu_{\text{par}^{j_0+1}(u')}([0, i]) = 0$  and  $\mu_{\text{par}^{j_0}(u')}([0, i]) > 0$ , we deduce from (4.1.2) (applied to  $u = \text{par}^{j_0+1}(u')$ ) that  $\lambda_{\text{par}^{j_0}(u')} = 0$ , and so the right-hand side of (5.1.12) vanishes. This means that (5.1.12) is again valid. Finally, if  $j_0 = n$ , then by (5.1.1) we have

$$\lambda_{u|u'}^{(i)} = \prod_{j=0}^{n-1} \lambda_{\text{par}^j(u')} \sqrt{\frac{\mu_{\text{par}^j(u')}([0, i])}{\mu_{\text{par}^{j+1}(u')}([0, i])}} = \lambda_{u|u'} \sqrt{\frac{\mu_{u'}([0, i])}{\mu_u([0, i])}},$$

which completes the proof of (5.1.12). Now we show that

$$(5.1.13) \quad \lim_{i \rightarrow \infty} \langle S_\lambda^n e_u, S_{\lambda^{(i)}}^n e_u \rangle = \|S_\lambda^n e_u\|^2, \quad u \in V, n \in \mathbb{Z}_+.$$

Indeed, it follows from Lemma 2.3.1(ii) and (5.1.12) that

$$\langle S_\lambda^n e_u, S_{\lambda^{(i)}}^n e_u \rangle = \sum_{u' \in \text{Chi}^{(n)}(u)} \lambda_{u|u'} \overline{\lambda_{u|u'}^{(i)}}$$



$$= \frac{1}{\sqrt{\mu_u([0, i])}} \sum_{u' \in \text{Chi}^{(n)}(u)} |\lambda_{u|u'}|^2 \sqrt{\mu_{u'}([0, i])}, \quad u \in V, n \in \mathbb{Z}_+, i \geq \kappa_u.$$

By applying Lebesgue's monotone convergence theorem for series, (5.1.7) and Lemma 2.3.1(iii), we obtain (5.1.13). Since

$$\|S_{\lambda}^n e_u - S_{\lambda^{(i)}}^n e_u\|^2 = \|S_{\lambda}^n e_u\|^2 + \|S_{\lambda^{(i)}}^n e_u\|^2 - 2 \operatorname{Re} \langle S_{\lambda}^n e_u, S_{\lambda^{(i)}}^n e_u \rangle$$

we infer (5.1.11) from (5.1.10) and (5.1.13). Clearly (5.1.11) implies (2.3.8).

We conclude this section with a general criterion for subnormality of weighted shifts on directed trees written in terms of determinacy of Stieltjes moment sequences.

**THEOREM 5.1.3.** *Let  $S_{\lambda}$  be a weighted shift on a directed tree  $\mathcal{T}$  with weights  $\lambda = \{\lambda_v\}_{v \in V^\circ}$  such that  $\mathcal{E}_V \subseteq \mathcal{D}^\infty(S_{\lambda})$ . Assume that  $\{\|S_{\lambda}^{n+1} e_u\|^2\}_{n=0}^\infty$  is a determinate Stieltjes moment sequence for every  $u \in V$ . Then the following conditions are equivalent:*

- (i)  $S_{\lambda}$  is subnormal,
- (ii)  $\{\|S_{\lambda}^n e_u\|^2\}_{n=0}^\infty$  is a Stieltjes moment sequence for every  $u \in V$ ,
- (iii) there exist a system  $\{\mu_u\}_{u \in V}$  of Borel probability measures on  $\mathbb{R}_+$  and a system  $\{\varepsilon_u\}_{u \in V}$  of nonnegative real numbers that satisfy (4.1.2) for every  $u \in V$ .

**PROOF.** (i)  $\Rightarrow$  (ii) Use Proposition 4.1.1.

(ii)  $\Rightarrow$  (iii) Employ Lemma 4.1.4.

(iii)  $\Rightarrow$  (i) Apply Theorem 5.1.1. □

Regarding Theorem 5.1.3, note that by Proposition 4.1.1, Lemma 4.2.2 (iv) and (2.4.1) each of the conditions (i), (ii) and (iii) implies that  $\{\|S_{\lambda}^{n+1} e_u\|^2\}_{n=0}^\infty$  is a Stieltjes moment sequence for every  $u \in V$ .

**5.2. Nonzero weights.** As pointed out in [17, Proposition 5.1.1] bounded hyponormal weighted shifts on directed trees with nonzero weights are always injective. It turns out that the same conclusion can be derived in the unbounded case (with almost the same proof). Recall that a densely defined operator  $S$  in  $\mathcal{H}$  is said to be *hyponormal* if  $\mathcal{D}(S) \subseteq \mathcal{D}(S^*)$  and  $\|S^* f\| \leq \|S f\|$  for all  $f \in \mathcal{D}(S)$ . It is well-known that subnormal operators are hyponormal (but not conversely) and that hyponormal operators are closable and their closures are hyponormal (we refer the reader to [30, 19] for more information on this subject).

**PROPOSITION 5.2.1.** *Let  $\mathcal{T}$  be a directed tree with  $V^\circ \neq \emptyset$ . If  $S_{\lambda}$  is a hyponormal weighted shift on  $\mathcal{T}$  whose all weights are nonzero, then  $\mathcal{T}$  is leafless. In particular,  $S_{\lambda}$  is injective and  $V$  is infinite and countable.*

**PROOF.** Suppose that, contrary to our claim,  $\text{Chi}(u) = \emptyset$  for some  $u \in V$ . We deduce from [17, Corollary 2.1.5] and  $V^\circ \neq \emptyset$  that  $u \in V^\circ$ . Hence, by [17, Propositions 3.1.3 and 3.4.1], we have

$$|\lambda_u|^2 = \|S_{\lambda}^* e_u\|^2 \leq \|S_{\lambda} e_u\|^2 = \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 = 0,$$

which is a contradiction. Since each leafless directed tree is infinite, we deduce from [17, Propositions 3.1.7 and 3.1.10] that  $S_{\lambda}$  is injective and  $V$  is infinite and countable. This completes the proof. □

The sufficient condition for subnormality of weighted shifts on directed trees stated in Theorem 5.1.1 takes the simplified form for weighted shifts with nonzero weights. Indeed, if a weighted shift  $S_{\lambda}$  on  $\mathcal{T}$  with nonzero weights satisfies the assumptions of Theorem 5.1.1, then, by assertions (ii) and (iii) of Lemma 4.2.1,  $\varepsilon_v = 0$  for every  $v \in V^\circ$ . Hence, by applying Theorem 5.1.1, we get.

**THEOREM 5.2.2.** *Let  $S_{\lambda}$  be a weighted shift on a directed tree  $\mathcal{T}$  with nonzero weights  $\lambda = \{\lambda_v\}_{v \in V^\circ}$  such that  $\mathcal{E}_V \subseteq \mathcal{D}^\infty(S_{\lambda})$ . Then  $S_{\lambda}$  is subnormal provided that one of the following two conditions holds:*

- (i)  $\mathcal{T}$  is rootless and there exists a system  $\{\mu_v\}_{v \in V}$  of Borel probability measures on  $\mathbb{R}_+$  which satisfies the following equality for every  $u \in V$ ,

$$(5.2.1) \quad \mu_u(\sigma) = \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 \int_{\sigma} \frac{1}{s} d\mu_v(s), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),$$

- (ii)  $\mathcal{T}$  has a root and there exist  $\varepsilon \in \mathbb{R}_+$  and a system  $\{\mu_v\}_{v \in V}$  of Borel probability measures on  $\mathbb{R}_+$  which satisfy (5.2.1) for every  $u \in V^\circ$ , and

$$\mu_{\text{root}}(\sigma) = \sum_{v \in \text{Chi}(\text{root})} |\lambda_v|^2 \int_{\sigma} \frac{1}{s} d\mu_v(s) + \varepsilon \delta_0(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+).$$

**5.3. Quasi-analytic vectors.** Let  $S$  be an operator in a complex Hilbert space  $\mathcal{H}$ . We say that a vector  $f \in \mathcal{D}^\infty(S)$  is a *quasi-analytic* vector of  $S$  if

$$\sum_{n=1}^{\infty} \frac{1}{\|S^n f\|^{1/n}} = \infty \quad (\text{convention: } \frac{1}{0} = \infty).$$

Denote by  $\mathcal{Q}(S)$  the set of all quasi-analytic vectors. Note that (cf. [42, Section 9])

$$(5.3.1) \quad S(\mathcal{Q}(S)) \subseteq \mathcal{Q}(S).$$

In general,  $\mathcal{Q}(S)$  is not a linear subspace of  $\mathcal{H}$  even if  $S$  is essentially selfadjoint (see [33]; see also [32] for related matter).

We now show that the converse of the implication in Proposition 4.1.1 holds for weighted shifts on directed trees having sufficiently many quasi-analytic vectors, and that within this class of operators subnormality is completely characterized by the existence of a consistent system of probability measures.

**THEOREM 5.3.1.** *Let  $S_{\lambda}$  be a weighted shift on a directed tree  $\mathcal{T}$  with weights  $\lambda = \{\lambda_v\}_{v \in V^\circ}$  such that  $\mathcal{E}_V \subseteq \mathcal{Q}(S_{\lambda})$ . Then the following conditions are equivalent:*

- (i)  $S_{\lambda}$  is subnormal,
- (ii)  $\{\|S_{\lambda}^n e_u\|^2\}_{n=0}^{\infty}$  is a Stieltjes moment sequence for every  $u \in V$ ,
- (iii) there exist a system  $\{\mu_v\}_{v \in V}$  of Borel probability measures on  $\mathbb{R}_+$  and a system  $\{\varepsilon_v\}_{v \in V}$  of nonnegative real numbers that satisfy (4.1.2) for every  $u \in V$ .

**PROOF.** (i) $\Rightarrow$ (ii) Apply Proposition 4.1.1.

(ii) $\Rightarrow$ (iii) Fix  $u \in V$  and set  $t_n = \|S_{\lambda}^{n+1} e_u\|^2$  for  $n \in \mathbb{Z}_+$ . By (2.4.1), the sequence  $\{t_n\}_{n=0}^{\infty}$  is a Stieltjes moment sequence. Since  $e_u \in \mathcal{Q}(S_{\lambda})$ , we infer from (5.3.1) that  $S_{\lambda} e_u \in \mathcal{Q}(S_{\lambda})$ , or equivalently that  $\sum_{n=1}^{\infty} t_n^{-1/2n} = \infty$ . Hence, by the Carleman criterion for determinacy of Stieltjes moment sequences (cf. [36, Theorem 1.11]), the Stieltjes moment sequence  $\{t_n\}_{n=0}^{\infty} = \{\|S_{\lambda}^{n+1} e_u\|^2\}_{n=0}^{\infty}$  is determinate. Now applying Lemma 4.1.4 yields (iii).

(iii) $\Rightarrow$ (i) Employ Theorem 5.1.1. □

Using [42, Theorem 7], one can prove a version of Theorem 5.3.1 in which the class of quasi-analytic vectors is replaced by the class of analytic ones. Since the former class is larger<sup>3</sup>, we see that “analytic” version of Theorem 5.3.1 is weaker than Theorem 5.3.1 itself. To the best of our knowledge, Theorem 5.3.1 is the first result of this kind; it shows that the unbounded version of Lambert’s characterization of subnormality happens to be true for operators that have sufficiently many quasi-analytic vectors.

The following result, which is an immediate consequence of Theorem 5.3.1, provides a new characterization of subnormality of bounded weighted shifts on directed trees written in terms of consistent systems of probability measures. It may be thought of as a complement to Theorem 4.1.2.

**COROLLARY 5.3.2.** *Let  $S_{\lambda} \in \mathcal{B}(\ell^2(V))$  be a weighted shift on a directed tree  $\mathcal{T}$  with weights  $\lambda = \{\lambda_v\}_{v \in V^\circ}$ . Then  $S_{\lambda}$  is subnormal if and only if there exist a system  $\{\mu_v\}_{v \in V}$  of Borel probability measures on  $\mathbb{R}_+$  and a system  $\{\varepsilon_v\}_{v \in V}$  of nonnegative real numbers that satisfy (4.1.2) for every  $u \in V$ .*

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<sup>3</sup> In general, the class of analytic vectors of an operator  $S$  is essentially smaller than the class of quasi-analytic vectors of  $S$  even for essentially selfadjoint operators  $S$  (cf. [32]).

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